Direct Solution of the "Three-Moments Equation"

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One hundred years ago Clapeyron noticed that the bending moments at three consecutive supports are connected by an invariable relation. When the spans are equal, the corresponding equation and the most useful boundary conditions can be written in the form

$$M_{n+1} + 4M_n + M_{n-1} = -G_n \tag{1}$$

$$M_0 = M_N = 0 \qquad n = 0, 1, \ldots, N \tag{2}$$

where M_n and G_n are moments and loading terms at consecutive supports. With help of the symmetrical operator

$$A_{nm} = \tfrac{1}{2}[|n - m| - (n + m) + (2nm/N)] \tag{3}$$

(introduced by the author), it is possible to express the exact solution of the boundary problem (1) and (2) in the form

$$M = (\mathbf{I} - 6\mathbf{A})^{-1}\mathbf{A}G \tag{4}$$

The operator $\mathbf{A}$ is nonsingular; thus

$$(\mathbf{I} - 6\mathbf{A})^{-1} = \sum_{\alpha=0}^{\infty} 6^\alpha \mathbf{A}^\alpha \tag{5}$$

and

$$M = -\frac{1}{6}\left(\mathbf{I} - \sum_{\alpha=0}^{\infty} 6^\alpha \mathbf{A}^\alpha\right)G \tag{6}$$

But for symmetrical tensor $\mathbf{A}$, one can write

$$\mathbf{A} = (1/\lambda_i)\mathbf{P}_{(i)}$$

and

$$\mathbf{A}^\alpha = [1/(\lambda_i)^\alpha]\mathbf{P}_{(i)} \qquad i = 0, 1, \ldots, N \tag{7}$$

where λ_i are eigenvectors, roots of the equation

$$|\mathbf{I} - \lambda\mathbf{A}| = 0$$

and

$$P_{(i)n,m} = C_{(i)n}C_{(i)m}$$

are projection tensors associated with orthogonal directions $C_{(i)}$ called normalized eigenvectors of $\mathbf{A}$.

Hence, introducing (7) into (5) yields

$$\sum_{\alpha=0}^{\infty} 6^\alpha \mathbf{A}^\alpha = \sum_{\alpha=0}^{\infty}\left(\frac{6}{\lambda_i}\right)^\alpha \mathbf{P}_{(i)}$$

When $(\mathbf{I} - 6\mathbf{A})^{-1}$ exist, then the series $\sum_{\alpha=0}^{\infty} (6/\lambda_i)^\alpha$ is convergent to $\lambda_i/(\lambda_i - 6)$ and

$$\sum_{\alpha=0}^{\infty} 6^\alpha \mathbf{A}^\alpha = \sum_{i=0}^{N} \frac{\lambda_i}{\lambda_i - 6}\mathbf{P}_{(i)}$$

The solution of the "three-moments equation" is therefore

$$M = -\frac{1}{6}\left(\mathbf{I} - \sum_{i=0}^{N} \frac{\lambda_i}{\lambda_i - 6}\mathbf{P}_{(i)}\right)G \tag{8}$$

Because of the boundary conditions (2), the first and last member of the series in (8) can be neglected, and the solution can be written as follows:

$$M = -\frac{1}{6}\left(\mathbf{I} - \sum_{i=1}^{N-1} \frac{\lambda_i}{\lambda_i - 6}\mathbf{P}_{(i)}\right)G \tag{9}$$

When, for example $N = 3$, then $\lambda_1 = 1$, $\lambda_2 = 3$, and

$$C_{(1)} = \begin{vmatrix} \dfrac{1}{2^{1/2}} \\ \dfrac{1}{2^{1/2}} \end{vmatrix} \qquad C_{(2)} = \begin{vmatrix} \dfrac{1}{2^{1/2}} \\ -\dfrac{1}{2^{1/2}} \end{vmatrix}$$

thus Eq. (8) takes the form

$$\begin{vmatrix} M_1 \\ M_2 \end{vmatrix} = -\frac{1}{6}\left(\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \frac{1}{5}\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} - \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}\right)\begin{vmatrix} G_1 \\ G_2 \end{vmatrix} \tag{10}$$

Assuming uniform load on each span and length of the span equal l, the load terms are $G_1 = G_2 = \frac{1}{2}pl^2$, and one therefore has

$$\begin{vmatrix} M_1 \\ M_2 \end{vmatrix} = -\frac{1}{6}\begin{vmatrix} \frac{8}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{8}{5} \end{vmatrix}\begin{vmatrix} (pl^2/2) \\ (pl^2/2) \end{vmatrix}$$

in agreement with results of the theory of elasticity.

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A Similar Solution of the Turbulent, Free-Convection, Boundary Layer Problem for an Electrically Conducting Fluid in the Presence of a Magnetic Field

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The free-convection boundary layer problem for an electrically conducting fluid in the presence of a magnetic field has been considered in Refs. 1–4 for the case of laminar flow. The purpose of this paper is to investigate the analogous problem for turbulent flow.

The physical model considered below is similar to that which was employed by Sparrow.[2] A magnetic field is impressed across a vertical plate that is kept at a constant temperature T_w in an electrically conducting fluid of ambient temperature T_e and conductivity σ.

If the usual[3] MHD, free-convection, boundary layer simplifications are adopted, the integrated momentum and energy equations can be written as follows:[3]

$$\frac{d}{dx}\int_0^\delta u^2 dy = g\beta\int_0^\delta \theta dy - \frac{\tau_w}{\rho} - \frac{1}{\rho}\int_0^\delta \sigma B_0 u dy \tag{1}$$

$$q_w = g\rho C_p \frac{d}{dx}\int_0^\delta u\theta dy \tag{2}$$

The notation used here is as usually employed in free-convection boundary layer analysis.

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In the absence of a magnetic field, Eckert[5] has shown that the $\frac{1}{7}$-power law for forced-convection turbulent flow can be applied to obtain expressions for u, θ, τ_w, and q in the free-convection case. Moffatt[6] applied the $\frac{1}{7}$-power law to obtain these expressions in forced-convection turbulent flow in the presence of a magnetic field. In accord with Eckert and Moffatt, let

$$u = u_e(y/\delta)^{1/7}[1 - (y/\delta)]^4 \quad (3)$$

$$\theta = \theta_w[1 - (y/\delta)^{1/7}] \quad (4)$$

$$\tau_w = 0.0225\,\rho u_e^2(\nu/u_e\delta)^{1/4} \quad (5)$$

$$q_w = 0.0225\,g\rho C_p u_e \theta_w(\nu/u_e\delta)^{1/4}Pr^{-2/3} \quad (6)$$

$$u_e = C_1 x^m \quad (7)$$

$$\delta = C_2 x^n \quad (8)$$

in the momentum and the energy equations and obtain

$$0.0523C_1^2C_2(2m + n)x^{2m+n-1} = 0.125g\beta\theta_w C_2 x^n - 0.130(\sigma B_0^2/\rho)C_2C_1x^{n+m} - 0.0225\nu^{1/4}C_1^{7/4}C_2^{-1/4}x^{2m-1/4(m+n)} \quad (9)$$

$$0.0366C_2C_1(n + m)x^{n+m-1} = 0.0225C_1^{3/4}C_2^{-1/4}\nu^{1/4}\rho^{-2/3}x^{m-(1/4)(m+n)} \quad (10)$$

By imposing the condition that B_0 must vary as $x^{-1/4}$, it is possible to solve for m and n by equating the exponents of x. Performing this operation, one gets $m = \frac{1}{2}$, $n = \frac{7}{10}$. With these values, Eqs. (9) and (10) now can be solved for C_1 and C_2, and, with the expressions obtained for C_1 and C_2, Eqs. (7) and (8) can be combined with Eq. (6) and arranged in dimensionless form as suggested by Cole:[7]

$$Nux = 0.002Pr^{1/3}Ra^{2/5}[1.27M + (1.62M^2 + 2.25Pr^{2/3} + 4.5)^{1/2}]^{-4/5} \quad (11)$$

Where Nu, Pr, Ra are the Nusselt, Prandtl, and Rayleigh numbers, respectively, and M is the magnetic parameter defined the same as in Ref. 3:

$$M = B_0^2x^{1/2}\sigma/\rho(g\beta\theta_w)^{1/2}$$

In the absence of a magnetic field, $M = 0$, and the reduction in heat transfer $(Nux)_w/(Nux)_0$ becomes

$$\frac{(Nux)_w}{(Nux)_0} = \left[\frac{(4.5 + 2.25Pr^{2/3})^{1/2}}{1.27M + (1.62M^2 + 4.5 + 2.25Pr^{2/3})^{1/2}}\right]^{4/5} \quad (12)$$

Figure 1 gives a comparison of the magnetic effect on the reduction in heat transfer between the laminar case[3] and the turbulent case derived here. It can be seen that for a given magnetic field much greater reduction in heat transfer is obtained in the turbulent case.

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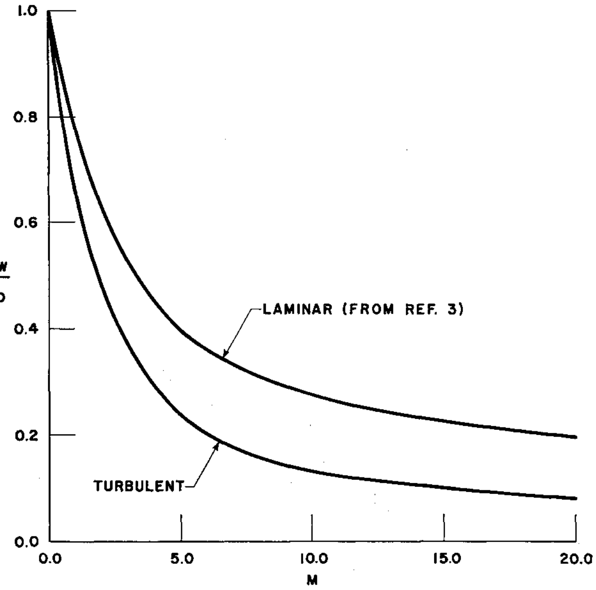

Fig. 1 Reduction in heat transfer $(Nux)w/(Nux)_0$ vs magnetic parameter M for laminar and turbulent flow ($Pr = 0.72$)

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Hypersonic Wake Transition

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Recent data taken at Massachusetts Institute of Technology[1] have yielded estimates of the transition behavior for the near wakes of blunt bodies at high M_∞ and sharp bodies for $M_\infty<10$. Representative values from these data, plotted as transition distance from the estimated wake neck position vs freestream Reynolds number, are shown in Fig. 1. The solid line in the figure was taken directly from Ref. 1. In Fig. 2, the same data are plotted vs local Reynolds number, $R_{f,d} = \rho_f U_f d/\mu_f$. For the sphere data, the fluid properties were evaluated both by using computed values at the axis for an inviscid flow and also by using values at the edge of the turbulent wake (computed via the method of Ref. 2). The solid line is the authors' best estimate of the true laminar values. Several interesting features are exhibited by these and other unpublished data:

1) At high Reynolds numbers the transition distance for the cone seems to "stick" at a fixed distance from the body, whereas for the sphere the transition location continues to move toward the wake neck as the Reynolds number is increased.

2) Transition distance appears to vary linearly with $R_{f,d}$ for both the sphere and the cone in the region $x_{tr}/d < 30$,

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